Math 10B with Professor Stankova
Worksheet, Discussion #17; Thursday, 3/15/2018
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1 Recursion Equations

1.1 Concepts

1. A **homogeneous** recursion does not include any extra constants (e.g. $a_n = a_{n-1} + a_{n-2}$) and a **nonhomogeneous** recursion contains one (e.g. $a_n = a_{n-1} + 4$). The **order** of a recursion equation is the "farthest" back the relation goes. For instance, the order of $a_n = a_{n-1} + a_{n-3}$ is 3 because we need the term 3 terms back (a_{n-3}).

   The general solution of a first order equation $a_n = a_{n-1} + d$ is $a_n = a_0 + nd$.

   In order to solve a linear homogeneous we can replace the equation with its characteristic polynomial. For instance, the characteristic polynomial of $a_n = 2a_{n-1} + a_{n-2}$ is $\lambda^2 = 2\lambda + 1$. Then if $\lambda_1, \ldots, \lambda_k$ are roots of this polynomial, then the general form of the solution is $a_n = C_1\lambda_1^n + \cdots + C_k\lambda_k^n$.

   The Δ operator takes in a series and spits out a new one. By definition, we have that $\Delta a_n = a_{n+1} - a_n$. This is done to change linear non-homogeneous equations into homogeneous ones.

1.2 Examples

2. Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ with $a_1 = 5, a_2 = 13$.

   Solution: This is a homogeneous equation and so we can solve this using the characteristic equation. The characteristic polynomial is $\lambda^2 = 5\lambda - 6$ or $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$. Thus, the general form of the solution has $a_n = c_1 2^n + c_2 3^n$. Now to solve for these constants, we plug in our initial conditions. We have that $a_1 = 2c_1 + 3c_2 = 5$ and $a_2 = 4c_1 + 9c_2 = 13$. Solving gives us $c_1 = c_2 = 1$ so the solution is $a_n = 2^n + 3^n$.

3. Solve the recurrence relation $a_n = 2a_{n-1} + 1$ with $a_0 = 0$.

   Solution: This is a non-homogeneous equation but we can homogenize it by subtracting this from the previous term which is $a_{n-1} = 2a_{n-2} + 1$. Doing so gives us $\Delta a_{n-1} = 2\Delta a_{n-2}$. We see that $a_1 = 1$ and hence $\Delta a_0 = 1$ which means that we can solve to get $\Delta a_n = 2^n$. Thus, we have that $a_{n+1} - a_n = 2^n$ or $a_{n+1} = a_n + 2^n$. In general, we see that $a_n = 2^0 + 2^1 + \cdots + 2^{n-1} = 2^n - 1$, which is the solution.

1.3 Problems

4. **TRUE** False We are not given an easy formula to plug in to solve linear non-homogeneous recursion equations.

5. **TRUE** False If a_n, b_n are two solutions to a linear homogeneous equation, then $a_n + b_n$ is also an equation.

Solution: If our recurrence was $a_n = 2a_{n-1}$ for instance, then $(a_n + b_n) = (2a_{n-1} + 2b_{n-1}) = 2(a_{n-1} + b_{n-1})$ showing that their sum is also a solution.

6. **TRUE** False If a_n is a solution to a linear homogeneous equation, then ca_n is also a solution for any constant c.

7. True **FALSE** If a_n, b_n are two solutions to a linear non-homogeneous equation, then $a_n + b_n$ is also an equation.

Solution: If our recurrence was $a_n = a_{n-1} + 1$ for instance, then $a_n = n$ is a solution and $b_n = n$ is also a solution but $a_n + b_n = 2n$ is not.

8. True **FALSE** If a_n is a solution to a linear non-homogeneous equation, then ca_n is also a solution for any constant c.

Solution: If our recurrence was $a_n = a_{n-1} + 1$ for instance, then $a_n = n$ is a solution but $2a_n = 2n$ is not.

9. Verify that $a_n = \binom{n}{5}$ is a solution to $a_n = \frac{n}{n-5} a_{n-1}$.

Solution: We plug in $\binom{n}{5}$ to get $\binom{n}{5} \stackrel{?}{=} \frac{n}{n-5}\binom{n-1}{5} = \frac{n}{n-5}\frac{(n-1)!}{5!(n-6)!} = \frac{n!}{5!(n-5)!} = \binom{n}{5}$ so this is indeed a solution.

10. Solve the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ with $a_0 = 3$ and $a_1 = 2$.

Solution: The characteristic polynomial is $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0$. Thus the general form is $a_n = c_1 4^n + c_2(-1)^n$. Plugging in our initial conditions gives $c_1 + c_2 = 3$ and $4c_1 - c_2 = 2$ which gives $c_1 = 1$ and $c_2 = 2$. So the answer is $a_n = 4^n - 2 \cdot (-1)^n$.

11. Find A, B such that $a_n = An + B$ is a solution to the recurrence relation $2a_n = a_{n-1} + 2a_{n-2} + n$.

> **Solution:** We plug it in to get $2An + 2B = A(n-1) + B + 2(A(n-2) + B) + n = 3An + n - 5A + 3B$. Hence $2A = 3A + 1$ and $A = -1$ while $2B = -5A + 3B$ so $B = 5A = -5$. Hence $a_n = -n - 5$.

1.4 Extra Problems

12. Verify that $a_n = n^2$ is a solution to $a_n = a_{n-1} + 2n - 1$.

> **Solution:** We plug in n^2 to get $n^2 \stackrel{?}{=} (n-1)^2 + 2n - 1 = n^2$ so this is indeed a solution.

13. Solve the recurrence relation $a_n = 4a_{n-1} + 5a_{n-2}$ with $a_0 = 3$ and $a_1 = 3$.

> **Solution:** The characteristic polynomial is $\lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0$. Thus the general form is $a_n = c_1 5^n + c_2(-1)^n$. Plugging in our initial conditions gives $c_1 + c_2 = 3$ and $5c_1 - c_2 = 3$ which gives $c_1 = 1$ and $c_2 = 2$. So the answer is $a_n = 5^n - 2 \cdot (-1)^n$.

14. Find A, B such that $a_n = An + B$ is a solution to the recurrence relation $3a_n = a_{n-1} + 3a_{n-2} + n + 5$.

> **Solution:** We plug it in to get $3An + 3B = A(n-1) + B + 3(A(n-2) + B) + n + 5 = 4An + n - 7A + 4B + 5$. Hence $3A = 4A + 1$ and $A = -1$ while $3B = -7A + 4B + 5$ so $B = 7A - 5 = -12$. Hence $a_n = -n - 12$.

2 Differential Equations

2.1 Concepts

15. The **order** of a differential equation is the highest derivative that appears in the equation. For instance, the equation $y''' + \sqrt{y'} = t^2 y$ is third order.

    A problem of the form $y' = f(t, y)$ and $y(0) = y_0$ is called an **initial value problem (IVP)**. There is a theorem that tells us when a solution to this problem exists. It says that if f is continuous, then for every choice of y_0, the solution **exists** in a time interval $[0, T)$ for some $0 < T \leq \infty$. But, the solution may not exist everywhere and it is not guaranteed to be unique.

If in addition f satisfies the **Lipschitz** condition (that $|f(t, y) - f(t, z)| \leq C|y - z|$) for some constant C and all y, z), then the solution is **unique** and exists for all $t \geq 0$. For instance $f(y) = y^2$ does not satisfy the Lipschitz condition because there is no constant such that $|y^2 - 0^2| \leq C|y - 0| = C|y|$ for all y. Effectively this is saying that f does not grow or shrink faster than a linear function.

2.2 Examples

16. Bacteria grows at a rate $N' = 0.05N$ where time is measured in hours. If initially there were 1000 cells, how many cells will there be in 10 hours?

Solution: We guess that a solution is of the form Ce^{rt}. Solving gives $rCe^{rt} = 0.05Ce^{rt}$ and hence $r = 0.05$ so $N = Ce^{0.05t}$. We know that $N(0) = Ce^0 = C = 1000$. So in 10 hours, we have $N(10) = 1000e^{0.05(10)} = 1000e^{0.5}$ cells.

2.3 Problems

17. **TRUE** False For an IVP, the function f may be continuous everywhere but still the solution does not exist everywhere.

18. True **FALSE** We guaranteed that the IVP $y' = ty^2$, $y(0) = 0$ has a unique solution.

Solution: In this case, $f(t, y) = ty^2$ which grows faster in the y variable than a linear function. So, the solution will blow up and we are not guaranteed that it is unique.

19. **TRUE** False We guaranteed that the IVP $y' = t^2y$, $y(0) = 0$ has a unique solution.

Solution: In this case, $f(t, y) = t^2y$ which is linear in y. So, the solution will be unique.

20. Solve the IVP $y' = te^t$ with $y(0) = 0$.

Solution: We solve by integrating te^t. We do this using integration by parts. Let $u = t$ and $dv = e^t dt$ so $du = dt$ and $v = e^t$. Integrating gives $\int te^t dt = te^t - \int e^t dt = te^t - e^t + C$. Now we solve for our initial condition that $y(0) = 0e^0 - e^0 + C = C - 1 = 0$ so $C = 1$. Thus the solution is $te^t - e^t + 1$.

21. Verify that $y = te^t + 1$ is a solution to $y'' - 2y' = 1 - y$.

Solution: We plug this into the equation $y'' - 2y' = 1 - y$ to get $(te^t + 2e^t) - 2(te^t + e^t) \stackrel{?}{=} 1 - (te^t + 1) = -te^t$ and $LHS = -te^t + 2e^t - 2e^t = -te^t = RHS$ so the sides agree and this is a solution.

2.4 Extra Problems

22. Solve the IVP $y' = \frac{1}{t \ln t}$ with $y(e) = 0$.

Solution: We find the integral using u substitution. Let $u = \ln t$ and then $y = \int \frac{1}{t \ln t} dt = \frac{du}{u} = \ln|u| + C + \ln(\ln t) + C$. Plug in our initial condition that $y(e) = 0$ to get $\ln(\ln e) + C = \ln 1 + C = C = 0$ so $y = \ln(\ln t)$ is our solution.

23. Verify that $y = 2e^{1/(2t)}$ is a solution to $2t^2 y' + y = 0$.

Solution: Plug this in to get $2t^2 y' + y$ to get $2t^2 (2 \frac{-2}{(2t)^2} e^{1/(2t)}) + 2e^{1/(2t)} = -2e^{1/(2t)} + 2e^{1/(2t)} = 0$ so this is a solution.

3 True/False Review

24. True **FALSE** To find $p(B|A)$, it suffices to know just $p(A|B)$ and how to apply Bayes' Theorem.

Solution: We also need to know $P(B)$ and $P(A|\bar{B})$.

25. **TRUE** False Among other things, the proof of Bayes' Theorem for finding $p(B|A)$ depends on being able to split the probability $p(A)$ as a sum probabilities $p(A \cap B)$ and $p(A \cap \overline{B})$, and then further rewrite these as products of certain other probabilities.

26. True **FALSE** The extra shortcut formula $p(B|A) = \frac{1}{1+\frac{p(A|\overline{B})\cdot p(\overline{B})}{p(A|B)\cdot p(B)}}$ works in one particular case when the standard formula for $p(B|A)$ in Bayes' Theorem fails.

> **Solution:** It is the opposite. The standard formula works in a case when this shortcut does not, which is when $P(A \cap B) = 0$.

27. True **FALSE** If a winner in a bicycle race tests positive for steroids, and this test has a very high "True Positive" rate and hence a very low "False positive" rate, then we should take away the winning cup from the athlete because it is extremely likely that he/she has used steroids.

> **Solution:** The actual answer could be a lot different from just the true positive rate.

28. **TRUE** False Error 1 in Hypothesis Testing (reject the null-hypothesis that the person is healthy when the person is actually healthy) is analogous to Testing positive for steroids (event T), yet not having used steroids (event $\overline{S}$); in other words, the significance α corresponds to $p(T \cap \overline{S})$.

29. **TRUE** False Error 2 in Hypothesis Testing (keep the null-hypothesis that the person is healthy but the person is, in fact, sick) is analogous to Testing negative for steroids (event $\overline{T}$), yet having used steroids (event S); in other words, the power of a test $1 - \beta$ corresponds to $1 - p(\overline{T} \cap S)$.

30. True **FALSE** To partition a set Ω into a disjoint union of subsets $B_1, B_2, \ldots, B_n$, means that the intersection of these sets is empty; i.e., $B_1 \cap B_2 \cap \cdots \cap B_n = \emptyset$.

> **Solution:** We need the pairwise intersections $B_i \cap B_j$ to be empty as well.

31. True **FALSE** Two disjoint events could be independent, but two independent events can never be disjoint.

> **Solution:** If one event is the empty set, then it is disjoint and independent with any other event.

32. True **FALSE** If a fair coin comes up Heads six times in a row, it is more likely that it will come up Tails than Heads on the 7th flip.

Solution: If it is fair, the flips are independent.

33. **TRUE** False Contrary to how we may use the word "dependent" in everyday life; e.g., event A could be dependent on event B, yet event B may not be dependent on event A; in math "dependent" is a symmetric relation; i.e., A is dependent with B if and only B is dependent with A.

34. **TRUE** False If A and B are independent events, their complements are also bound to be independent, and to prove this we need a general argument since an example is not sufficient here.

35. True **FALSE** If A and B are independent events, $\overline{A}$ and B may fail to be independent, but to prove this we need just one counterexample, not a general proof.

Solution: They are also independent.

36. True **FALSE** If any pair of events among $A_1, A_2, ..., A_n$ are independent, then all events are independent.

37. True **FALSE** A random variable (RV) on a probability space (Ω, P) is a function $X : \Omega \to \mathbb{R}$ that satisfies certain rules and is related to the probability function P.

Solution: A RV is **any** function to $\mathbb{R}$ and not at all related to the probability function P. The PMF of this RV is related to P though.

38. **TRUE** False A RV X could be the only source of data for an outcome space Ω and hence could be very useful in understanding better X 's domain.

Solution: One reason we introduced random variables is because it is sometimes hard to understand all of Ω and we can only look at it through X.